University "Politehnica" of Bucharest
Department of Mathematics & Informatics

PhD Thesis Abstract

Iteration Theory, Continuous Optimization
and non-Newtonian Calculus

Author: Alia Shani Hassan Kurdi

Supervised by Prof. Dr. habil. Mihai Postolache

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**Keywords:** Nearly asymptotically nonexpansive mapping, three-step iteration scheme, fixed point, strong convergence, $\Delta$-convergence, CAT($k$) space, strictly pseudocontractive mapping, iteration scheme with perturbed mapping, common fixed point, best proximity point, almost contraction, metric spaces, $C$-class function, multiple objective programming, multitime multiobjective variational problem, vector variational-like inequalities, invex functionals, properly efficient solutions, $b$-multiplivative metric space.
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PhD Abstract

The content presented in this Thesis is mainly oriented on Iteration Theory (Chapter 1 and Chapter 2), Continuous Optimization (Chapter 3 and Chapter 4) and non-Newtonian Calculus (Chapter 5). The study is motivated by the current research in mathematics, while the results herein are published in selective mathematical journals.

Fixed point theory in CAT($k$) space was first studied by Kirk in [51]. His works were followed by a series of new works by many authors, mainly focusing on CAT(0) spaces: see, e.g., Abkar and Eslamian [3], Chang et al. [21]. It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT($k$) space with $k \leq 0$ since any CAT($k$) space is a CAT($m$) space for every $m \geq k$; see Bridson and Haefliger [16].

The class of asymptotically nonexpansive mapping was introduced by Goebel and Kirk [30] in 1972, as an important generalization of the class of nonexpansive mapping and they proved that if $C$ is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self mapping of $C$ has a fixed point.

There are number of papers dealing with the approximation of fixed points of asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces using modified Mann, Ishikawa and three-step iteration processes: see, e.g., Liu [55], Saluja [68], Schu [72], Shahzad and Udomene [74], Tan and Xu [78], Xu and Noor [83]; see also, Ceng et al. [18].

The concept of $\Delta$-convergence in a general metric space was introduced by Lim [54]. In 2008, Kirk and Panyanak [52] used the notion of $\Delta$-convergence introduced by Lim [54] to prove in the CAT(0) space and analogous of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [24] obtained $\Delta$-convergence theorems for the Picard, Mann and Ishikawa iterations in a CAT(0) space. Since then, the existence problem and the $\Delta$-convergence problem of iterative sequences to a fixed point for nonexpansive mapping, asymptotically nonexpansive mapping, nearly asymptotically nonexpansive mapping, asymptotically nonexpansive mapping in the intermediate sense, asymptotically quasi-nonexpansive mapping in the intermediate sense, total asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mapping through Picard, Mann [56], Ishikawa [40], modified Agarwal et al. [6] have been rapidly developed in the framework of CAT(0) space and many papers have appeared in this direction: see, e.g., Abbas et al. [1], Dhompongsa and Panyanak [24], Khan and Abbas [45], Kumam et al. [53].
In Chapter 1, **Three-step iteration process in a CAT\((k)\) space**, we establish \(\Delta\)-convergence and strong convergence of modified three-step iteration process which contains modified S-iteration process for a class of mappings which is wider than that of asymptotically nonexpansive mappings in CAT\((k)\) spaces. Our results extend and improve the corresponding results of Abbas et al. [1], Dhompongsa and Panyanak [24], Khan and Abbas [45] and many other results of this direction. Some examples are also provided to show that these results are more general than the well-known results existing in literature.

Motivated and inspired by [6] and some others, we introduce a new iteration scheme as follows.

**Algorithm A.** The sequence \(\{x_n\}\) defined by \(x_1 \in K\) and

\[
\begin{align*}
z_n &= (1 - \gamma_n)x_n \oplus \gamma_n T^n x_n \\
y_n &= (1 - \beta_n)x_n \oplus \beta_n T^n z_n, \\
x_{n+1} &= (1 - \alpha_n)T^n x_n \oplus \alpha_n T^n y_n, \quad n \geq 1,
\end{align*}
\]

where \(\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\gamma_n\}_{n=1}^{\infty}\) are appropriate sequences in \((0,1)\), is called modified three-step iterative sequence. Iteration scheme in Algorithm A is independent of the modified Noor iteration, modified Ishikawa iteration and modified Mann iteration schemes. If \(\gamma_n = 0\) for all \(n \geq 1\), then our Algorithm reduces to the modified S-iteration.

Iteration procedures in fixed point theory are lead by the considerations in summability theory. For example, if a given sequence converges, then we do not look for the convergence of the sequence of its arithmetic means. Similarly, if the sequence of Picard iterates of any mapping \(T\) converges, then we don’t look for the convergence of other iteration procedures.

The three-step iterative approximation problems were studied extensively by Noor [58], Glowinsky and Le Tallec [29], and Haubruge et al. [36]. The three-step iterations lead to highly parallelized algorithms under certain conditions. They are also a natural generalization of the splitting methods for solving partial differential equations. It has been shown [29] that three-step iterative scheme gives better numerical results than the two step and one step approximate iterations. Thus we conclude that three step scheme plays an important and significant role in solving various problems, which arise in pure and applied sciences. These facts motivated us to study a class of three-step iterative schemes in the setting of CAT\((k)\) spaces with \(k > 0\).

Our results in this chapter are: Theorem 1.1, Theorem 1.2, Theorem 1.3, Theorem 1.4, Lemma 1.5, Lemma 1.6, Corollary 1.1, Corollary 1.2, Corollary 1.3. They are published
in [67] (G.S. Saluja, M. Postolache, A. Kurdi, Convergence of three-step iterations for nearly asymptotically nonexpansive mappings in CAT(k) spaces, J. Inequal. Appl. 2015, Art. No. 156 (IF 0.630)).

Next, we list our main results.

**Theorem 0.1.** Let $k > 0$ and $(X, d)$ be a complete CAT(k) space with $\text{diam}(X) = \frac{\pi / 2 - \varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi / 2)$. Let $K$ be a nonempty closed convex subset of $X$ and let $T : K \to K$ be a continuous nearly asymptotically nonexpansive mapping. Then $T$ has a fixed point.

**Theorem 0.2.** Let $k > 0$ and $(X, d)$ be a complete CAT(k) space with $\text{diam}(X) = \frac{\pi / 2 - \varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi / 2)$. Let $K$ be a nonempty closed convex subset of $X$ and let $T : K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mapping. If $\{x_n\}$ is an AFPS for $T$ such that $\Delta\text{-lim}_{n \to \infty} x_n = z$, then $z \in K$ and $z = Tz$.

**Lemma 0.1.** Let $k > 0$ and $(X, d)$ be a complete CAT(k) space with $\text{diam}(X) = \frac{\pi / 2 - \varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi / 2)$. Let $K$ be a nonempty, closed and convex subset of $X$ and let $T : K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} (\eta(T^n) - 1) < \infty$. Let $\{x_n\}$ be a sequence in $K$ defined by Algorithm A. Then $\lim_{n \to \infty} d(x_n, p)$ exists for each $p \in F(T)$.

**Lemma 0.2.** Let $k > 0$ and $(X, d)$ be a complete CAT(k) space with $\text{diam}(X) = \frac{\pi / 2 - \varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi / 2)$. Let $K$ be a nonempty, closed and convex subset of $X$ and let $T : K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} (\eta(T^n) - 1) < \infty$. Let $\{x_n\}$ be a sequence in $K$ defined by Algorithm A. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $\lim \inf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$, $\lim \inf_{n \to \infty} \beta_n (1 - \beta_n) > 0$ and $\lim \inf_{n \to \infty} \gamma_n (1 - \gamma_n) > 0$. Then $\lim_{n \to \infty} d(x_n, T x_n) = 0$.

**Theorem 0.3.** Let $k > 0$ and $(X, d)$ be a complete CAT(k) space with $\text{diam}(X) = \frac{\pi / 2 - \varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi / 2)$. Let $K$ be a nonempty, closed and convex subset of $X$ and let $T : K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} (\eta(T^n) - 1) < \infty$. Let $\{x_n\}$ be a sequence in $K$ defined by Algorithm A. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $\lim \inf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$, $\lim \inf_{n \to \infty} \beta_n (1 - \beta_n) > 0$ and $\lim \inf_{n \to \infty} \gamma_n (1 - \gamma_n) > 0$. Then $\{x_n\}$ $\Delta$-converges to a fixed point of $T$.
Theorem 0.4. Let $k > 0$ and $(X, d)$ be a complete CAT($k$) space with $\text{diam}(X) = \frac{\pi/2 - \varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$. Let $K$ be a nonempty, closed and convex subset of $X$ and let $T: K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} \left( \eta(T^n) - 1 \right) < \infty$. Let $\{x_n\}$ be a sequence in $K$ defined by Algorithm A. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $\lim \inf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$, $\lim \inf_{n \to \infty} \beta_n (1 - \beta_n) > 0$ and $\lim \inf_{n \to \infty} \gamma_n (1 - \gamma_n) > 0$. Suppose that $T^m$ is semi-compact for some $m \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of $T$.


Motivated and inspired by the above described contribution, we propose a new implicit iteration scheme with perturbed mapping, to approximate fixed point of a finite family of $\kappa_i$-strictly pseudocontractive self-mappings $\{T_i\}_{i=1}^{N}$ as we will explain below.

Let $E$ be a real Banach space and $G: E \to E$ be a perturbed mapping which is both $\lambda'$-strictly pseudocontractive and $\delta$-strongly accretive with $\lambda' + \delta \geq 1$. For an arbitrary initial point $x_0 \in E$, the sequence $\{x_n\}$ is generated by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) \left[ \beta_n (x_n - \lambda G(x_n)) + \gamma_n \sum_{i=1}^{N} \mu_i T_i x_n \right],$$

where $\{\mu_i\}_{i=1}^{N}$ is a sequence of weights satisfying $\sum_{i=1}^{N} \mu_i = 1$, $\{\alpha_n\} \subset (0, 1)$, $\{\gamma_n\} \subset (0, 1]$, $\{\beta_n\} \subset [0, 1)$ and $\lambda \in [0, 1)$.

In Chapter 2, named Implicit iteration for pseudocontractive mappings, we establish some strong and weak convergence theorems for a finite family of strictly pseudocontractive mappings using implicit iteration scheme (1). We illustrate our results on concrete examples and numerically compute the fixed point. The results obtained in this
chapter improve and extend the results of Xu and Ori [84], Ceng et al. [20], Chen et al. [19] and some other results in this direction.

Our results in this chapter are: Proposition 2.1, Lemma 2.8, Lemma 2.9, Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4, Example 2.1. They are published in [81] (B.S. Thakur, R. Dewangan, A. Kurdi, Implicit iteration scheme with numerical analysis for a finite family of pseudocontractive mappings, U. Politeh. Buch. Ser. A 79(2017), No. 1, 11-24 (IF 0.365)).

Among the results in this chapter we emphasize:

**Proposition 0.1.** Let $E$ be a smooth Banach space and $G: E \to E$ be both $\lambda'$-strictly pseudocontractive and $\delta$-strongly accretive with $\lambda' + \delta \geq 1$. Then $S_\lambda = (I - \lambda G): E \to E$ is a pseudocontractive mapping, for $0 \leq \lambda < 1$.

**Lemma 0.3.** Let $E$ be a real smooth Banach space $E$, $G: E \to E$ be both $\lambda'$-strictly pseudocontractive and $\delta$-strongly accretive with $\lambda' + \delta \geq 1$ and $T_i: E \to E$ be a finite family of $\kappa_i$-strictly pseudocontractive mappings, where $i \in \{1, 2, \ldots, N\}$ such that $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences satisfying the conditions $0 < a \leq \alpha_n \leq b < 1$, $0 < a \leq \gamma_n \leq 1$, $\beta_n + \gamma_n = 1$ and $\sum_{n=1}^{\infty} \beta_n < \infty$. Let $\{x_n\}$ be a sequence generated by (1). Then

(a) $\lim_{n \to \infty} \|x_n - p\|$ exists, for all $p \in \mathcal{F}$;

(b) $\lim_{n \to \infty} d(x_n, \mathcal{F})$ exists, where $d(x_n, \mathcal{F}) = \inf_{p \in \mathcal{F}} \|x_n - p\|$.

**Lemma 0.4.** Let $E$ be a real reflexive and smooth Banach space, $G$ and $T_i$ are as in Lemma 0.3 and all conditions of Lemma 0.3 are satisfied. Let $S = \sum_{i=1}^{N} \mu_i T_i: E \to E$ be a $\kappa$-strictly pseudocontractive mapping. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in E$ by (1). Then $\lim_{n \to \infty} \|Sx_n - x_n\| = 0$.

Now, we introduce our weak convergence results.

**Theorem 0.5.** Let $E$ be a uniformly convex Banach space satisfying Opial’s condition. Let $G$, $T_i$ and $S$ are as in Lemma 0.4 and all conditions of Lemma 0.4 are satisfied. Then the sequence $\{x_n\}$ generated by (1) converges weakly to a member of $\mathcal{F}$.

**Theorem 0.6.** Let $E$, $G$, $T_i$ and $S$ are as in Lemma 0.3 and all conditions of Lemma 0.3 are satisfied. Then the sequence $\{x_n\}$ generated by (1) converges strongly to a member of $\mathcal{F}$ if and only if $\lim \inf_{n \to \infty} d(x_n, \mathcal{F}) = 0$. 
To introduce the strong convergence results, we recall the following definitions:

**Definition 0.1** ([73]). A mapping $T: E \to E$ with $F(T) \neq \emptyset$ is said to satisfy condition (A) on $E$ if there exists a nondecreasing function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > r$ for all $r \in (0, \infty)$ such that for all $x \in E$,

$$\|x - Tx\| \geq f(d(x, F(T))),$$

where $d(x, F(T)) = \inf_{x^* \in F(T)} \|x - x^*\|$.

**Definition 0.2** ([80]). A finite family $T_i: E \to E$ of self mappings, where $i = \{1, 2, \ldots, N\}$ with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ is said to satisfy condition (BS) on $E$ if there exist $f$ and $d$ as in Definition 0.1, such that

$$\|x - Sx\| \geq f(d(x, F)) \quad \text{for all } x \in E$$

where $S = \sum_{i=1}^{N} \mu_i T_i$ and $\{\mu_i\}_{i=1}^{N}$ is a sequence of positive number such that $\sum_{i=1}^{N} \mu_i = 1$.

**Theorem 0.7.** Let $E$, $G$, $T_i$ and $S$ be as in Lemma 0.4 and all conditions of Lemma 0.4 are satisfied and let finite family of $T_i$ satisfies condition (BS). Then the sequence $\{x_n\}$ generated by (1) converges strongly to a member of $F$.

Let $E$ be a Banach space. A mapping $T: E \to E$ is said to be semicompact, if for any bounded sequence $\{x_n\}$ in $E$ such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to x^* \in K$ as $j \to +\infty$.

We now establish a strong convergence result using semicompact condition.

**Theorem 0.8.** Let $E$, $G$, $T_i$ and $S$ be as in Lemma 0.4 and all conditions of Lemma 0.4 are satisfied and let $S = \sum_{i=1}^{N} \mu_i T_i$ be semicompact. Then the sequence $\{x_n\}$ generated by (1) converges strongly to a member of $F$.

The chapter is completed with numerical experiments of several case studies.

The problem of finding the minimum distance between two subsets of a metric space, the so called proximity point problem, has been developed in various settings: partially ordered metric spaces in [22] by Choudhury et al., Banach spaces in [8] by Al-Thafai and Shahzad and in [27] by Eldered and Veeramani, hyperconvex metric spaces and in Hilbert spaces in [50] by Kirk et al., Menger probabilistic metric spaces in [41] by Jamali and Vaezpour. The notion of proximal pointwise contraction and results regarding the
existence of a best proximity point on a pair of weakly compact convex subset of a Banach space are obtained by Eldered and Veeramani in [26]. Convergence and existence results of best proximity points for diverse classes of mappings are studied by: Karpagam and Agrawal [46], Sankar Raj and Veeramani [70]. In [71], Sankar Raj stated a fixed point theorem for weakly contractive nonselfmappings based on the notion of (P)-property.

In Chapter 3, titled **Proximal point in metric space with (P)-property**, we consider a problem of global optimization in the context of a complete metric space with (P)-property. For this purpose, we utilize our class of generalized almost contractive non-self-maps. More accurately, using the concept of a C-class function, three types of generalized almost \((f,\psi,\varphi,\theta)\)-contractions are defined. By the (P)-property, existence and uniqueness of some best proximity points are stated and proved in this new setting. The presented results are a natural continuation of those of Shatanawi and Pitea [76], Choudhury et al [22], Sankar Raj [71].

Our results in this chapter are: Definition 3.6, Theorem 3.2, Corollary 3.1, Corollary 3.2, Definition 3.7, Theorem 3.3, Definition 3.8, Theorem 3.4. They are published in [9] (A.H. Ansari, W. Shatanawi, A. Kurdi, G. Maniu, Best proximity points in complete metric spaces with (P)-property via C-class functions, J. Math. Anal. 7(2016), No. 6, 54-67.

Let \(A\) and \(B\) be nonempty subsets of a metric space. In the sequel, the following two sets associated to \(A\) and \(B\) are considered.

\[
A_0 = \{a \in A : d(a, b) = d(A, B), \text{ for some } b \in B\},
\]

\[
B_0 = \{b \in B : d(a, b) = d(A, B), \text{ for some } a \in A\}.
\]

As usual, \(d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}\).

**Definition 0.3** ([69]). Let \(A\) and \(B\) be two nonempty subsets of a metric space \((X, d)\). An element \(u \in A\) is said to be a best proximity point of the nonselfmapping \(T : A \to B\) if it satisfies the condition

\[
d(u, Tu) = d(A, B).
\]

**Definition 0.4** ([85]). Let \((A, B)\) be a pair of nonempty subsets of a metric space \((X, d)\) with \(A_0 \neq \emptyset\). Then, the pair \((A, B)\) is said to have the weak (P)-property if, for each \(x_1, x_2 \in A\), and \(y_1, y_2 \in B\), the following implication holds

\[
\begin{align*}
&d(x_1, y_1) = d(A, B) \\
&d(x_2, y_2) = d(A, B)
\end{align*}
\]

\[\Rightarrow\]

\[
d(x_1, x_2) \leq d(y_1, y_2).
\]
If we replace relation $d(x_1, x_2) \leq d(y_1, y_2)$ by $d(x_1, x_2) = d(y_1, y_2)$ we obtain a less general notion, that of a pair endowed with the $(P)$-property, see [71].

In [76], Shatanawi and Pitea studied a best proximity point result with regard to an almost contraction for a pair of sets endowed with the weak $(P)$-property.

**Definition 0.5** ([14]). A map $\varphi: [0, +\infty) \rightarrow [0, +\infty)$ is called a $c$-comparison function if it satisfies:

1. $\varphi$ is a monotone increasing,
2. $\sum_{n=0}^{+\infty} \varphi^n(t)$ converges for all $t \geq 0$.

By replacing the second condition by $\lim_{n \to +\infty} \varphi^n(t) = 0$, $\forall t \in [0, +\infty)$, we get the notion of comparison function, more general than the one of $c$-comparison function. It is known that if $\varphi$ is a comparison function, then $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$.

In the following, denote $[0, +\infty) \times [0, +\infty) \times [0, +\infty) \times [0, +\infty)$ by $[0, +\infty)^4$.

Let $\Theta$ be the set of all continuous functions $\theta: [0, +\infty)^4 \rightarrow [0, +\infty)$ such that $\theta(0, t, s, u) = 0$ for all $t, s, u \in [0, +\infty)$ and $\theta(t, s, 0, u) = 0$ for all $t, s, u \in [0, +\infty)$.

Examples of functions in $\Theta$ are available in [69].

In Shatanawi and Pitea [76], the following type of contraction was used.

**Definition 0.6** ([76]). Let $\varphi$ be a comparison function, and $\theta \in \Theta$. Mapping $T: A \rightarrow B$ is called a generalized almost $(\varphi, \theta)$-contraction if, for each $x, y \in A$,

$$d(Tx, Ty) \leq \varphi(d(x, y)) + \theta(d(y, Tx) - d(A, B), d(x, Ty) - d(A, B),
\begin{align*}
d(x, Tx) - d(A, B),
d(y, Ty) - d(A, B)
\end{align*}$$

By using thin type of contraction function, the following theorem is proved.

**Theorem 0.9** ([76]). Consider $A$ and $B$ two closed subsets of a complete metric space $(X, d)$ for which $A_0$ is nonempty. Let $T: A \rightarrow B$ be a mapping which satisfies the following conditions:

1) $T$ is a generalized almost $(\varphi, \theta)$-contraction;
2) $TA_0 \subseteq B_0$;
3) The pair $(A, B)$ has the weak $(P)$-property.

Then, there exists a unique best proximity point of $T$, $x^* \in A$. 
Let $\Psi$ be the set of all continuous functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

$(\psi_1)$ $\psi$ is continuous and strictly increasing;

$(\psi_2)$ $\psi(t) = 0$ if and only if $t = 0$.

Let $\Phi_u$ be the set of all continuous functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

$(\phi_1)$ $\varphi$ is continuous.

$(\phi_2)$ $\varphi(t) > 0$ if $t > 0$ and $\varphi(0) \geq 0$.

Let $\Phi$ be the set of all lower continuous functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, such $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) < t$ for all $t > 0$.

In 2014, A.H. Ansari [11] introduced the concept of $C$-class functions. By using this concept we can generalize many fixed point theorems in literature.

**Definition 0.7** ([11]). Let $f : \mathbb{R}_+^2 \to \mathbb{R}$ be a continuous mapping. $f$ is called a $C$-class function if it satisfies the following conditions:

$(C_1) : f(s,t) \leq s$, for all $(s,t) \in \mathbb{R}_+^2$.

$(C_2) : f(s,t) = s$, implies that $s = 0$, or $t = 0$, for all $(s,t) \in \mathbb{R}_+^2$.

Note that if $f$ is a $C$-class function, then $f(0,0) = 0$.

We denote the set of all the $C$-class functions by $\mathcal{C}$; see [10]. In [11] we find examples of functions which are elements of $\mathcal{C}$.

Our aim in Chapter 3 is to introduce and prove best proximity point theorems for a more general case. For this instance, we introduce the notion of a generalized almost $(f, \psi, \varphi, \theta)$-contraction of the first type, as follows

**Definition 0.8.** Let $\psi \in \Psi$, $\varphi \in \Phi_u$, $f \in \mathcal{C}$, and $\theta \in \Theta$. The mapping $T : A \to B$ is called a generalized almost $(f, \psi, \varphi, \theta)$-contraction of the first type if, for each $x, y \in A$,

$$
\psi(d(Tx, Ty)) \leq f(\psi(d(x, y)), \varphi(d(x, y))) + \\
\theta(d(y, Tx) - d(A, B), d(x, Ty) - d(A, B), \\
d(x, Tx) - d(A, B), d(y, Ty) - d(A, B)).
$$
Our first result is

**Theorem 0.10.** Consider $A$ and $B$ two closed subsets of a complete metric space $(X, d)$ for which $A_0$ is nonempty. Let $T: A \to B$ be a mapping which satisfies the following conditions:

1) $T$ is a generalized almost $(f, \psi, \varphi, \theta)$-contraction of the first type;
2) $TA_0 \subseteq B_0$;
3) the pair $(A, B)$ has the weak $(P)$-property.

Then, there exists a unique best proximity point of $T$, $x^* \in A$.

Let us take the particular case of $f(s, t) = kt$, where $k \in (0, 1)$, and

$$\theta: [0, +\infty)^4 \to [0, +\infty), \quad \theta(t_1, t_2, t_3, t_4) = L \min\{t_1, t_2, t_3, t_4\},$$

for some $L \geq 0$. We obtain the following corollary.

**Corollary 0.1.** Let $A$ and $B$ be two closed subsets of a complete metric space $(X, d)$ for which $A_0$ is nonempty. Let $T: A \to B$ be a mapping which satisfies the following conditions:

1) $TA_0 \subseteq B_0$;
2) the pair $(A, B)$ has the weak $(P)$-property.

Suppose there exist $k \in (0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq kd(x, y) + L \min\{d(y, Tx) - d(A, B), d(x, Ty) - d(A, B), d(x, Tx) - d(A, B), d(y, Ty) - d(A, B)\}$$

holds for all $x, y \in A$. Then, there exists a unique best proximity point of $T$, $x^* \in A$.

Also, we have

**Corollary 0.2.** Let $A$ be a closed subset of a complete metric space $(X, d)$. Consider $T: A \to A$ be a mapping such that

$$\psi(d(Tx, Ty)) \leq f\left(\psi(d(x, y)), \varphi(d(x, y))\right) + \theta(d(y, Tx), d(x, Ty), d(x, Tx), d(y, Ty))$$

holds for all $x, y \in A$, where $\psi \in \Psi, \varphi \in \Phi_u, f \in C$. Then $T$ has a unique fixed point $u \in A$; that is $Tu = u$. 
**Definition 0.9.** Let $\psi \in \Psi$, $\varphi \in \Phi_u$, $f \in C$, and $\theta \in \Theta$. A mapping $T: A \to B$ is called a generalized almost $(f, \psi, \varphi, \theta)$-contraction of the second type if, for each $x, y \in A$,

$$
\psi(d(Tx, Ty)) \leq f\left( \varphi\left( \frac{d(x, Tx) + d(y, Ty)}{2} - d(A, B) \right) \right) + \theta\left( \frac{d(x, Ty) + d(y, Tx)}{2} - d(A, B) \right).
$$

Our second result is

**Theorem 0.11.** Consider $A$ and $B$ two closed subsets of a complete metric space $(X, d)$ for which $A_0$ is nonempty. Let $T: A \to B$ be a mapping which satisfies the following conditions:

1) $T$ is a generalized almost $(f, \psi, \varphi, \theta)$-contraction of the second type;
2) $TA_0 \subseteq B_0$;
3) the pair $(A, B)$ has the weak $(P)$-property.

Then, there exists a unique best proximity point of $T$, $x^* \in A$.

We will introduce now our third type of generalized almost $(f, \psi, \varphi, \theta)$, as follows.

**Definition 0.10.** Let $\psi \in \Psi$, $\varphi \in \Phi_u$, $f \in C$, and $\theta \in \Theta$. A mapping $T: A \to B$ is called a generalized almost $(f, \psi, \varphi, \theta)$-contraction of the third type if, for each $x, y \in A$,

$$
\psi(d(Tx, Ty)) \leq f\left( \varphi\left( \frac{d(x, Ty) + d(y, Tx)}{2} - d(A, B) \right) \right) + \theta\left( \frac{d(x, Tx) + d(y, Ty)}{2} - d(A, B) \right).
$$

By using this definition, the following result can be obtained.

**Theorem 0.12.** Consider $A$ and $B$ two closed subsets of a complete metric space $(X, d)$ for which $A_0$ is nonempty. Let $T: A \to B$ be a mapping which satisfies the following conditions:

1) $T$ is a generalized almost $(f, \psi, \varphi, \theta)$-contraction of the third type;
2) $TA_0 \subseteq B_0$;
3) the pair $(A, B)$ has the weak $(P)$-property.

Then, there exists a unique best proximity point of $T$, $x^* \in A$. 

In optimization problems, one investigates the best solutions from all feasible solutions. These kind of problems with two or more objective functions to be optimized simultaneously are called vector optimization problems. It is difficult to find unique solution of these type of problems because they rarely have feasible points that simultaneously maximize or minimize all the objectives. Hence, it is necessary to explore the concept of efficient solutions. Several scientists have devoted in this direction: please, see Borwein [15] and Kazmi [47].

The growing interest in optimization, asks for the generalization of convexity, as the concept of convexity does no longer suffices in real world problems. Initially, Hanson [35] generalized convex functions to introduce the concept of invexity. Other generalizations such as preinvex, univex, pseudoinvex, approximate convex functions are available: Khurana [48], Gupta et al. [33].

Variational inequalities have wide applications in science. Due to these extensive attentions vector variational inequality was developed and initially formulated by Giannessi [28]. Vector variational inequalities are efficient tool for the investigation of vector optimization problems because these inequalities ensure the existence of efficient solutions, under the condition of convexity or generalized convexity. For recent approaches, we refer to Ruiz-Garzón et al. [65], Al-Homidan and Ansari [7].

Variational and optimal control problems proved to be a very useful and powerful tool for engineering studies. Variational problems are divided into two categories: one is vector continuous-time problem and other is classical variational problem. Kim [49] formulated vector variational-type inequalities and demonstrated the relationships between these inequalities and vector continuous-time problems. Also, see: Postolache [64], Pitea and Antczak [62].

In Chapter 4, Vector variational inequalities in optimization we introduce the vector variational-like inequality (VVLI) with its weak formulation for multitime multiobjective variational problem (MVP). Moreover, we establish the relationships between the solutions of introduced inequalities and (properly) efficient solutions of MVP, involving the invexities of multitime functionals. Examples are provided to illustrate the results.

Our results in this chapter are: Theorem 4.1, Theorem 4.2, Theorem 4.3, Theorem 4.4, Theorem 4.5, Definition 4.4, Definition 4.5, Definition 4.6, Example 4.1, Example 4.2, Example 4.3, Example 4.4, Example 4.5. They are published in [42] (A. Jayswal, S. Singh, A. Kurdi, Multitime multiobjective variational problems and vector variational-like

Let $M$ and $N$ be Riemannian manifolds of the dimensions $m$, $n$, endowed with local coordinates $t = t^\alpha$, $\alpha = \{1, \ldots, m\}$ and $x = x^j$, $j = \{1, \ldots, n\}$, respectively. By using the product order relation on $\mathbb{R}^m$, the hyperparallelepiped $\beta_{t_o, t_1}$ in $\mathbb{R}^m$ with diagonal opposite points $t_o = (t_1^o, \ldots, t_m^o)$ and $t_1 = (t_1^1, \ldots, t_m^1)$ can be written as interval $[t_o, t_1]$. Further, let $\Gamma_{t_o, t_1}$ be a piecewise $C^1$-class curve joining the points $t_o$, $t_1$ and $J^1(M, N)$ be the first order jet bundle associated to $M$ and $N$. $C^\infty(\beta_{t_o, t_1}, N)$ denotes the space of functions $x : \beta_{t_o, t_1} \mapsto N$ of $C^\infty$-class with the norm

$$\|x\| = \|x\|_\infty + \sum_{\alpha=1}^m \|x_\alpha\|_\infty.$$ 

From now onwards let $X \subset C^\infty(\beta_{t_o, t_1}, N)$ unless otherwise specified. For any $n$-dimensional vectors $x$ and $y$, we use the usual convention for equalities and inequalities throughout the chapter.

Consider the following MVP:

\textbf{(MVP)} Minimize $\int_{\Gamma_{t_o, t_1}} f_\alpha(\pi_x(t))dt^\alpha = \left(\int_{\Gamma_{t_o, t_1}} f^1_\alpha(\pi_x(t))dt^\alpha, \ldots, \int_{\Gamma_{t_o, t_1}} f^p_\alpha(\pi_x(t))dt^\alpha\right)$ subject to $x(t) \in X$,

where $f_\alpha : J^1(M, N) \mapsto \mathbb{R}$, $i \in P = \{1, \ldots, p\}$ are closed 1-forms of $C^\infty$-class. Here, $\pi_x(t) = (t, x(t), x_\gamma(t))$, and $x_\gamma(t) = \frac{\partial x(t)}{\partial t^\gamma}$, $\gamma = \{1, \ldots, m\}$ are partial velocities.

The closeness conditions (complete integrability conditions) are

$$D_\alpha f^i_\beta = D_\beta f^i_\alpha, \; i \in P, \; \alpha = \beta = \{1, \ldots, m\} \text{ and } \alpha \neq \beta,$$

where $D_\alpha$ and $D_\beta$ are total derivatives.

Multitime multiobjective variational problems have an inevitable deal, finding (weakly, properly) efficient solutions from the set of all feasible solutions. In this chapter, we shall take advantage of the sense of Pitea and Antczak [62].

\textbf{Definition 0.11 ([62])}. A point $y(t) \in X$ is called an efficient solution of (MVP), if there exists no $x(t) \in X$ such that

$$\int_{\Gamma_{t_o, t_1}} f^i_\alpha(\pi_x(t))dt^\alpha - \int_{\Gamma_{t_o, t_1}} f^i_\alpha(\pi_y(t))dt^\alpha \leq 0, \; \forall i \in P,$$

with strict inequality for at least one $i$. 
**Definition 0.12 ([62]).** A point $y(t) \in X$ is called a weak efficient solution of (MVP), if there exists no $x(t) \in X$ such that
\[
\int_{\Gamma_{t_0,t_1}} f_i^*(\pi_x(t)) dt^\alpha - \int_{\Gamma_{t_0,t_1}} f_i^*(\pi_y(t)) dt^\alpha < 0, \ \forall \ i \in P.
\]

**Definition 0.13 ([62]).** A point $y(t) \in X$ is said to be a proper efficient solution of (MVP), if it is an efficient solution of (MVP) and if there exists a positive scalar $M$ such that for all $i \in P$,
\[
\int_{\Gamma_{t_0,t_1}} f_i^*(\pi_x(t)) dt^\alpha - \int_{\Gamma_{t_0,t_1}} f_i^*(\pi_y(t)) dt^\alpha \leq M \left( \int_{\Gamma_{t_0,t_1}} f_i^1(\pi_x(t)) dt^\alpha - \int_{\Gamma_{t_0,t_1}} f_i^1(\pi_y(t)) dt^\alpha \right),
\]
for some $j$ such that
\[
\int_{\Gamma_{t_0,t_1}} f_j^1(\pi_x(t)) dt^\alpha > \int_{\Gamma_{t_0,t_1}} f_j^1(\pi_y(t)) dt^\alpha,
\]
whenever, $x(t) \in X$ and
\[
\int_{\Gamma_{t_0,t_1}} f_i^1(\pi_x(t)) dt^\alpha < \int_{\Gamma_{t_0,t_1}} f_i^1(\pi_y(t)) dt^\alpha.
\]

Now, we can introduce the following VVLI and weak vector variational-like inequality (WVVLI), respectively, which will be used to ensure the existence of efficient solutions of considered MVP.

Let $f_i^1: J^1(\beta_{t_0,t_1}, N) \to \mathbb{R}$, $i \in P$ be closed 1-forms of $C_\infty$-class, $\eta: J^1(\beta_{t_0,t_1}, N) \times J^1(\beta_{t_0,t_1}, N) \to \mathbb{R}^n$ and $D_y$ is total derivative.

**VVLI** Find $y(t) \in X$ such that there exists no $x(t) \in X$, satisfying
\[
\left( \int_{\Gamma_{t_0,t_1}} \left[ \langle \eta(\pi_x(t), \pi_y(t)), \frac{\partial f_i^1}{\partial x}(\pi_y(t)) \rangle + \langle D_y \eta(\pi_x(t), \pi_y(t)), \frac{\partial f_i^1}{\partial x}(\pi_y(t)) \rangle \right] dt^\alpha, \ldots, \right.
\]
\[
\left. \int_{\Gamma_{t_0,t_1}} \left[ \langle \eta(\pi_x(t), \pi_y(t)), \frac{\partial f_i^p}{\partial x}(\pi_y(t)) \rangle + \langle D_y \eta(\pi_x(t), \pi_y(t)), \frac{\partial f_i^p}{\partial x}(\pi_y(t)) \rangle \right] dt^\alpha \right) \leq 0.
\]

**WVVLI** Find $y(t) \in X$ such that there exists no $x(t) \in X$, satisfying
\[
\left( \int_{\Gamma_{t_0,t_1}} \left[ \langle \eta(\pi_x(t), \pi_y(t)), \frac{\partial f_i^1}{\partial x}(\pi_y(t)) \rangle + \langle D_y \eta(\pi_x(t), \pi_y(t)), \frac{\partial f_i^1}{\partial x}(\pi_y(t)) \rangle \right] dt^\alpha, \ldots, \right.
\]
\[
\left. \int_{\Gamma_{t_0,t_1}} \left[ \langle \eta(\pi_x(t), \pi_y(t)), \frac{\partial f_i^p}{\partial x}(\pi_y(t)) \rangle + \langle D_y \eta(\pi_x(t), \pi_y(t)), \frac{\partial f_i^p}{\partial x}(\pi_y(t)) \rangle \right] dt^\alpha \right) < 0.
An example to show that vector variational-like inequality (VVLI), which we have introduced, is solvable at a point can be found in [42].

By keeping the view of definitions of invexities and generalized invexities, we introduce the following invexity and pseudo-invexity for the multitime functional, which will be applicable in proving our results.

Let \( g_\alpha : J^1(\beta_{t_0,t_1}, N) \rightarrow \mathbb{R} \), \( \eta : J^1(\beta_{t_0,t_1}, N) \times J^1(\beta_{t_0,t_1}, N) \rightarrow \mathbb{R}^n \) be closed \( 1 \)-form of \( C_\infty \)-class and vector valued function, respectively, with \( \eta(\pi_x(t), \pi_x(t)) = 0 \).

**Definition 0.14.** A functional \( \int_{\Gamma_{t_0,t_1}} g_\alpha(\pi_x(t))dt^\alpha \) is said to be (strictly) invex with respect to \( \eta \) at \( y(t) \in X \), if for all \( x(t) \in X \), \( (x(t) \neq y(t)) \), the following inequality holds:

\[
\int_{\Gamma_{t_0,t_1}} g_\alpha(\pi_x(t))dt^\alpha - \int_{\Gamma_{t_0,t_1}} g_\alpha(\pi_y(t))dt^\alpha(>) \geq \int_{\Gamma_{t_0,t_1}} \left[ \langle \eta(\pi_x(t), \pi_y(t)), \frac{\partial g_\alpha}{\partial x}(\pi_y(t)) \rangle + \langle D_\gamma \eta(\pi_x(t), \pi_y(t)), \frac{\partial g_\alpha}{\partial x_\gamma}(\pi_y(t)) \rangle \right] dt^\alpha.
\]

An example which numerically shows that there exists a functional, which follows the above introduced definition is given in [42].

**Definition 0.15.** A functional \( \int_{\Gamma_{t_0,t_1}} g_\alpha(\pi_x(t))dt^\alpha \) is said to be pseudo-invex with respect to \( \eta \) at \( y(t) \in X \), if for all \( x(t) \in X \), the following implication holds:

\[
\int_{\Gamma_{t_0,t_1}} \left[ \langle \eta(\pi_x(t), \pi_y(t)), \frac{\partial g_\alpha}{\partial x}(\pi_y(t)) \rangle + \langle D_\gamma \eta(\pi_x(t), \pi_y(t)), \frac{\partial g_\alpha}{\partial x_\gamma}(\pi_y(t)) \rangle \right] dt^\alpha \geq 0
\]

\[
\Rightarrow \int_{\Gamma_{t_0,t_1}} g_\alpha(\pi_x(t))dt^\alpha - \int_{\Gamma_{t_0,t_1}} g_\alpha(\pi_y(t))dt^\alpha \geq 0.
\]

Equivalently,

\[
\int_{\Gamma_{t_0,t_1}} g_\alpha(\pi_x(t))dt^\alpha - \int_{\Gamma_{t_0,t_1}} g_\alpha(\pi_y(t))dt^\alpha < 0
\]

\[
\Rightarrow \int_{\Gamma_{t_0,t_1}} \left[ \langle \eta(\pi_x(t), \pi_y(t)), \frac{\partial g_\alpha}{\partial x}(\pi_y(t)) \rangle + \langle D_\gamma \eta(\pi_x(t), \pi_y(t)), \frac{\partial g_\alpha}{\partial x_\gamma}(\pi_y(t)) \rangle \right] dt^\alpha < 0.
\]

Example for the above definition is given in [42].

**Definition 0.16.** Let \( S \) be a nonempty subset of \( X \). Then, \( S \) is said to be invex with respect to \( \eta \), if for all \( x(t), y(t) \in S \),

\[
y(t) + \lambda \eta(\pi_x(t), \pi_y(t)) \in S, \quad 0 \leq \lambda \leq 1.
\]
Now, we state the relations between the solutions of introduced VVLI and (weak, proper) efficient solutions of considered MVP.

Let \( f^i_\alpha : J^1(\beta_{t_0,t_1}, N) \mapsto \mathbb{R} \), \( i \in P \) be closed 1-forms of \( C_\infty \)-class.

**Theorem 0.13.** Let \( X \) be invex set with respect to \( \eta \) and for each \( i \in P \), the functional \( \int_{\Gamma_{t_0,t_1}} f^i_\alpha(\pi_x(t))dt^\alpha \) be Fréchet differentiable at \( y(t) \in X \). If \( y(t) \) is a proper efficient solution of (MVP), then it solves (VVLI).

In [42] are introduced examples illustrate the result established in the above theorem.

**Theorem 0.14.** For each \( i \in P \), let the functional \( \int_{\Gamma_{t_0,t_1}} f^i_\alpha(\pi_x(t))dt^\alpha \) be invex with respect to \( \eta \) and Fréchet differentiable at \( y(t) \in X \). If \( y(t) \) solves (VVLI), then it is an efficient solution of (MVP).

**Theorem 0.15.** Let \( X \) be invex set with respect to \( \eta \). If \( y(t) \in X \) is a weak efficient solution of (MVP), then it solves (WVVLI).

**Theorem 0.16.** For each \( i \in P \), let the functional \( \int_{\Gamma_{t_0,t_1}} f^i_\alpha(\pi_x(t))dt^\alpha \) be pseudo-invex with respect to \( \eta \) and Fréchet differentiable at \( y(t) \in X \). If \( y(t) \) solves (WVVLI), then it is a weak efficient solution of (MVP).

**Theorem 0.17.** Let \( X \) be invex set with respect to \( \eta \) and for each \( i \in P \), the functional \( \int_{\Gamma_{t_0,t_1}} f^i_\alpha(\pi_x(t))dt^\alpha \) be strictly invex with respect to the same \( \eta \) and Fréchet differentiable at \( y(t) \in X \). If \( y(t) \) is a weak efficient solution of (MVP), then it is an efficient solution of (MVP).

Between 1967 and 1972, Grossman and Katz [31] introduced the non-Newtonian calculus consisting of the branches of geometric, bigeometric, quadratic and biquadratic calculus etc. Also, Grossman extended this notion to the other fields in [32]. All these calculi can be described simultaneously within the framework of a general theory. We will use the name non-Newtonian to indicate any calculi other than the classical calculus. Every property in the classical calculus has an analogue in non-Newtonian calculus which is a methodology that allows one to have a different look at problems which can be investigated via calculus. In some cases, for example for wage-rate (in dollars, euro etc.) related problems, the use of bigeometric calculus which is a kind of non-Newtonian calculus is advocated instead of a traditional Newtonian one.
Recently, Bashirov et al. [13] have concentrated on the multiplicative calculus and have given results with applications corresponding to the well-known properties of derivatives and integrals in the classical calculus. Also, Uzer [82] extended the non-Newtonian calculus to the complex valued functions and was interested in the statements of some fundamental theorems and concepts of multiplicative complex calculus, and proved some analogies between the multiplicative complex calculus and classical calculus by theoretical and numerical examples. Further, Misirli and Gurefe introduced multiplicative Adams Bashforth-Moulton methods for differential equations in [57]. Some authors also worked on the classical sequence spaces and related topics by using non-Newtonian calculus: please, see Çakmak and Başar [17], Tekin and Başar [79]. Further, Kadak [43] and Kadak et al. [44] have determined matrix transformations between certain sequence spaces over the non-Newtonian complex field and generalized Runge-Kutta method via non-Newtonian differentiation.

Following Çakmak and Başar [17], in Chapter 5, with the title **New viewpoint on non-Newtonian calculus**, we construct the classical sequence spaces with respect to the multiplicative calculus and introduce the new notion of $b$-multiplicative metric space. Some required inequalities are presented in the sense of the multiplicative calculus, and the concepts of $*$metric and related examples are given. We introduce the corresponding results for the sequences concerning the convergent sequences of real numbers and prove some basic topological properties. By using the notion of $*$completeness, $*$limit and $*$convergence, other results are discussed in detail. By means of a new fixed point result, we give an application to a class of multiplicative integral equations.

Our results in this chapter are: Theorem 5.1, Theorem 5.2, Theorem 5.3, Definition 5.9, Example 5.1, Lemma 5.2, Theorem 5.4, Corollary 5.1, Theorem 5.5. Part of them are published in [34] (Y. Gurefe, U. Kadak, E. Misirli, A. Kurdi, A new look at the classical sequence spaces by the using multiplicative calculus, U. Politeh. Buch. Ser. A, 78(2016), No. 2, 9-20 (IF 0.365)).

According to [17], let $X$ be a non-empty set and $d^*: X \times X \to \mathbb{R}^+$ be a function such that for all $x, y, z \in X$, the following axioms hold:

(M1) $d^*(x, y) = 1$ if and only if $x = y$,

(M2) $d^*(x, y) = d^*(y, x)$,

(M3) $d^*(x, y) \leq d^*(x, z) \oplus d^*(z, y)$.

Then, the pair $(X, d^*)$ and $d^*$ are called a multiplicative metric space and a multiplica-
tive metric (shortly, *metric) on $X$, respectively.

**Theorem 0.18.** Let $(x_n)$ be a sequence in a multiplicative metric space $X = (X, d^*)$. Then the following holds:

(i) Every *convergent sequence in a multiplicative metric space is a Cauchy sequence.

(ii) Every Cauchy sequence is *bounded.

(iii) If the Cauchy sequence $(x_n)$ have a subsequence $(x_{n_k})$ which converges to $x_0$, then $x_n \rightarrow^* x_0$.

Now, consider $\omega = \{(x_n) \mid x_n \in M, x_n > 0\}$. We define the classical sets $\ell^*_\infty(M)$, $c^*(M)$, $c^*_0(M)$ and $\ell^*_p(M)$ consisting of the multiplicative bounded, convergent, null and absolutely $p$-summable sequence, as follows:

$$
\ell^*_\infty(M) := \left\{ x = (x_k) \in \omega : \sup_{k \in \mathbb{N}} d^*(x_k, 1) < \infty \right\},
$$

$$
c^*(M) := \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{R}, \lim_{k \to \infty} d^*(x_k, l) = 1 \right\},
$$

$$
c^*_0(M) := \left\{ x = (x_k) \in \omega : \lim_{k \to \infty} d^*(x_k, 1) = 1 \right\},
$$

$$
\ell^*_p(M) := \left\{ x = (x_k) \in \omega : \exp \left( \sum_{k=1}^{\infty} (\ln |x_k|)^p \right) < \infty \right\}, \quad (1 \leq p < \infty).
$$

**Theorem 0.19.** Define the distance function $d^*_\infty$ by

$$
d^*_\infty : \gamma(M) \times \gamma(M) \rightarrow \mathbb{R}^+, \quad d^*_\infty(x, y) = \sup \{ |x_k \otimes y_k|^* : k \in \mathbb{N} \},
$$

where $\gamma$ denotes any of the spaces $\ell^*_\infty$, $c^*$ and $c^*_0$, and $x = (x_k), y = (y_k) \in \gamma(M)$. Then, $(\gamma(M), d^*_\infty)$ is a *complete metric space.

**Theorem 0.20.** Let $x = (x_k), y = (y_k) \in \ell^*_p(M)$ be sequences, and

$$
d^*_p : \ell^*_p(M) \times \ell^*_p(M) \rightarrow \mathbb{R}^+ \quad d^*_p(x, y) = \exp \left\{ \sum_{k=1}^{\infty} \left( \ln \frac{|x_k|^*}{|y_k|^*} \right)^p \right\}^{\frac{1}{p}}.
$$

$(\ell^*_p(M), d^*_p)$ is a *complete metric space.

By using the ideas of Grossman and Katz [31], Bashirov et al. [13] defined the notion of multiplicative metric. Özavsar and Cevikel [60] investigated the multiplicative metric spaces along with their topological properties and proved some fixed point theorems
for contraction mappings of multiplicative metric spaces. Effective contribution in this
direction is due to: Abbas et al. [2], He et al. [38].

Czerwik [23] introduced the notion of $b$-metric space, which is a generalization of a
metric space. There are some fixed point theorems in $b$-metric spaces: Huang et al. [39],
Ozturk and Turkoglu [61], Shatanawi et al. [75].

In their elegant survey, Došenović et al. [25] show that the fixed point results for
various multiplicative contractions are in fact equivalent with the corresponding fixed
point results in (standard) metric spaces.

In the last section of Chapter 5, we introduced the notion of $b$-multiplicative metric
space and mentioned its topological properties. We proved a fixed point theorem for
mappings on $b$-multiplicative metric spaces endowed with a graph. As novel application,
we give an existence theorem for the solution of a class of Fredholm multiplicative integral
equations. We note that these fixed point results are new in the setting of $b$-metric space
endowed with a graph, as far as we know.

**Definition 0.17.** Let $X$ be a nonempty set, and $s \geq 1$ a given real number. A mapping
$m : X \times X \to [1, \infty)$ is called a $b$-multiplicative metric if the following conditions hold:

- $(m_1)$ $m(x, y) > 1$ for all $x, y \in X$ with $x \neq y$ and $m(x, y) = 1$ if and only if $x = y$;
- $(m_2)$ $m(x, y) = m(y, x)$ for all $x, y \in X$;
- $(m_3)$ $m(x, z) \leq m(x, y)^s \cdot m(y, z)^s$ for all $x, y, z \in X$.

The triplet $(X, m, s)$ is called a $b$-multiplicative metric space.

**Theorem 0.21.** Let $(X, m, s)$ be a complete $b$-multiplicative metric space endowed the
graph $G$ and let $f : X \to X$ be an edge preserving mapping such that for each $(x, y) \in E$,
we have

$$
m(fx, fy) \leq \max \{m(x, y), m(x, fx), m(y, fy), m(x, fy)^{\frac{1}{s}} \cdot m(y, fx)\}^{\kappa}
$$

where $\kappa \in [0, \frac{1}{s})$. Assume that the following conditions hold:

(i) there exists $x_0 \in X$ such that $(x_0, fx_0) \in E$;

(ii) a. $f$ is $G$-continuous;

or

b. for each sequence $\{x_n\} \subseteq X$ such that $(x_n, x_{n+1}) \in E$ and $x_n \to_b x$, then
$(x_n, x) \in E$ for each $n \in \mathbb{N}$. 

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Then $f$ has a fixed point.

**Corollary 0.3.** Let $(X, m, s)$ be a complete $b$-multiplicative metric space endowed the graph $G$ and let $f : X \to X$ be an edge preserving mapping such that for each $(x, y) \in E$, one of the following inequality hold:

(i) $m(fx, fy) \leq m(x, y)\kappa$;

(ii) $m(fx, fy) \leq m(x, fx)\kappa$;

(iii) $m(fx, fy) \leq m(y, fy)\kappa$;

(iv) $m(fx, fy) \leq \{m(x, fy)^{1/2} \cdot m(y, fx)\}^\kappa$.

where $\kappa \in [0, \frac{1}{s})$. Assume that the following conditions hold:

(i) there exists $x_0 \in X$ such that $(x_0, fx_0) \in E$;

(ii) a. $f$ is $G$-continuous;

or

b. for each sequence $\{x_n\} \subseteq X$ such that $(x_n, x_{n+1}) \in E$ and $x_n \to_b x$, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$.

Then $f$ has a fixed point.

Now, we introduce an application to Fredholm multiplicative integral equations. Let $X = C([a, b], \mathbb{R}_+)$, $a > 0$ and $\mathbb{R}_+ = (0, \infty)$, be the space of all positive, continuous real valued functions, endowed with the $b$-multiplicative metric

$$m(x, y) = \begin{cases} 
\sup_{t \in [a, b]} \left| \frac{x(t)}{y(t)} \right|^2 & \text{if } \frac{x(t)}{y(t)} > 1 \\
\sup_{t \in [a, b]} \left| \frac{y(t)}{x(t)} \right|^2 & \text{if } \frac{x(t)}{y(t)} < 1
\end{cases}$$

and graph $G = (V, E)$ such that $V = X$ and $E = \{(x, y) : x(t) \geq y(t), \forall t \in [a, b]\}$.

We give an existence theorem for the Fredholm multiplicative integral equation of the following type.

$$x(t) = \int_a^b K(t, s, x(s))ds, \ t, s \in [a, b]$$

where $K : [a, b] \times [a, b] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing function.
Theorem 0.22. Let $X = C([a,b], \mathbb{R}_+)$, $a > 0$, endowed with the graph $G$ and let the operator

$$F: X \to X, \quad Fx(t) = \int_a^b K(t, s, x(s)) \, ds$$

where $K: [a, b] \times [a, b] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing function. Assume that the following conditions hold:

(i) for each $t, s \in [a, b]$ and $x, y \in X$ with $(x, y) \in E$, there exists a constant $\eta > 0$ such that

$$\left| \frac{K(t, s, x(s))}{K(t, s, y(s))} \right| \leq \left( \frac{|x(s)|}{|y(s)|} \right)^{\eta};$$

(ii) the constant $\eta$ is such that $\eta < \frac{1}{2(b-a)}$;

(iii) there exists $x_0 \in X$ such that $(x_0, Fx_0) \in E$.

Then the integral equation (2) has at least one solution.

References


3. A. Abkar, M. Eslamian, Common fixed point results in CAT(0) spaces, Nonlinear Anal. 74(2011), No. 5, 1835-1840.


14. V. Berinde, Approximating fixed points of weak \( \phi \)-contractions using the Picard iteration, Fixed Point Theory Appl. 4 (2003), No. 2, 131-142.


54. T.C. Lim, Remarks on some fixed point theorems, Proc. Amer. Math. Soc. 60(1976), 179-182.


63. A. Pitea, M. Postolache, Duality theorems for a new class of multitime multiobjective variational problems, J. Glob. Optim. 54(2012), 47-58.


70. V. Sankar Raj, P. Veeramani, Best proximity pair theorems for relatively nonexpansive mappings, Appl. General Topolgy 10(2009), 21-28.


76. W. Shatanawi, A. Pitea, Best proximity point and best proximity coupled point in a complete metric space with (P)-property, Filomat 29(2015), 63-74.

77. W. Shatanawi, M. Postolache, Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces, Fixed Point Theory Appl. 2013, Art. No. 54.


82. A. Uzer, Multiplicative type complex calculus as an alternative to the classical calculus, Comput. Math. Appl. 60(2010), 2725-2737.

